Help Notes

27 octobre 2017

Epsilon definitions for a limit

Limits help to understand the behavior of a real function at a point. Several thing can happen :

1. you can take the limit at a point within the domain of definition
   (a) if your function behave well at this one point, we will say in the next section that the function is continuous meaning that taking the limit at this point is just the value of the function at this point ;
   (b) if your function behave a bit strangely at this point, then taking limits might be a way that you can understand what is going on around there : it could be that there is a jump going on at this point and approaching your point on the right or on the left is not the same story or it could be that your function goes crazy around the point, for instance, it explodes to ∞ or $-\infty$ or goes up and down without never getting nowhere.
2. if now you want to understand the behavior of a function when the values on the domain become either big in the positive number or in the negative numbers, there is no way to compute the function at every single point of the domain of definition if there is an infinite amount of points on this domain and no one even a computer could do this, as we will need an infinite amount of time to do this and no one has an infinite amount of time ahead of them. Still sometimes you can decide and explicitly compute what is happening for those values of x and we will explain this in a mathematical way in a moment.

When we write

$$lim_{x \to a^+} f(x) = L$$

it means when you approach a from the right with x values but never reaching it, the images of this x values which are $f(x)$ approach L.

When we write

$$lim_{x \to a^-} f(x) = L$$

it means when you approach a from the left with x values but never reaching it, the images of this x values which are $f(x)$ approach L.

When we write

$$lim_{x \to a} f(x) = L$$

it means when you approach a (no matter if you come from the right or from the left) with x values but never reaching it, the images of this x values which are $f(x)$ approach L.

Note that :

Theorem 0.1.

$$lim_{x \to a} f(x) = L$$

if and only if

$$lim_{x \to a^+} f(x) = L \text{ and } lim_{x \to a^-} f(x) = L$$

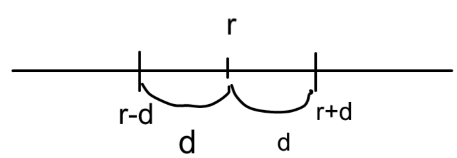
Also,

Theorem 0.2. *If* $lim_{x \to a^+} f(x) \neq lim_{x \to a^-} f(x) = L$ *then* $lim_{x \to a} f(x)$ *does not exist.*

BE AWARE, that since what you want to know is what is happening when you are very very close to a, it does not matter how your function behave when you are far from a. Your function could go crazy far from a and then calm down as you get close to a. This is very important to understand limit. When you take a limit, you try to get close enough to a so that from this point on, you are able to say something about your function and then once you are there and you can decide you see what is going on, trying to approach a closer and closer and closer.

Here, a and L could be either a real value or ∞ or $-\infty$.

Let's think what does it means for a point to be close to a real number r, we all agree I believe that it means that the distance lets call this distance d between those two points is very small. Well, we have two options either the point very close to r is on the right or on the left. Also the point on the left at a distance d is $r - d$ and the point of the right is $r + d$ as illustrated above.



So you get close to r as you make d become very very small. So when you are in the interval $(-r + d, r + d)$ and d is very very small, you sure to be close to r. Note that saying that $x \in (-r + d, r + d)$ is the same as saying $|x - r| < d$, remember that $|x - r|$ represent the distance between x and r.

So now we understand what it means to be very close to a real number. How about being very close to infinity ? Well, a point is very close to infinity if it is very very big, so if I take a big number and any number bigger than this number I am getting "close" in some sense to infinity how do I get closer just by taking a bigger number. So, if you choose a huge number M, if a number belongs to (M, ∞) you can see it as being "close to ∞" is you take a huger number M', a number belonging to (M', ∞) is closer to ∞ etc. So now, it is about making M bigger that you can get closer than infinity. Note that $x \in (M, \infty)$ is the same as $x > M$

How about being very close to - infinity ? Well, a point is very close to infinity if it is very very big in the negative, so if I take a big number in the negative (meaning a negative number whose absolute value is big) and any number bigger in the negative than this number

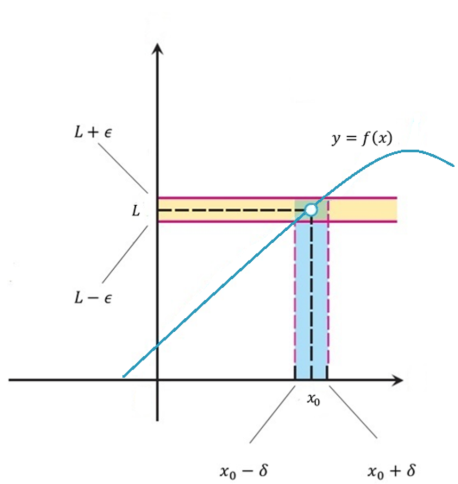
I am getting "close" in some sense to - infinity how do I get closer just by taking a bigger number in the negative. So, if you choose a huge number in the negative (meaning for instance -1000000000000000000), M, if a number belongs to $(-\infty, M)$ you can see it as being "close to $-\infty$" is you take a huger number in the negative M', a number belonging to $(-\infty, M')$ is closer to $-\infty$ etc. So now, it is about making M bigger in the negative that you can get closer than infinity. Note that $x \in (-\infty, M)$ is the same as $x < M$.

Now we understand what it mean to be close to anything.

Let us see how to write down properly the limit definition.

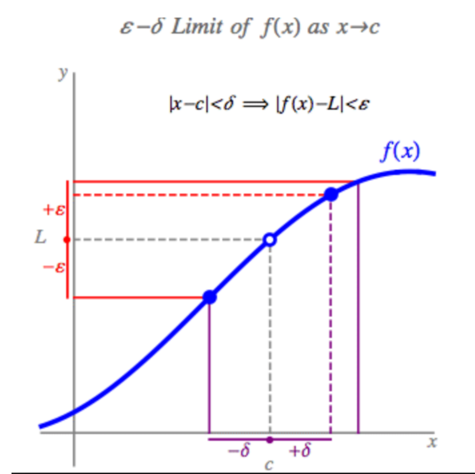
1. Suppose a and L are both real numbers. What does $lim_{x\to a} f(x) = L$ means? Well, as we just said it means that $f(x)$ can become as close as you want from L, if you take x very close to a. Which means there is points $f(x)$ close to L as close as I want to, which means I can make the distance between $f(x)$ and L is very small just by making the distance from x to a very small.

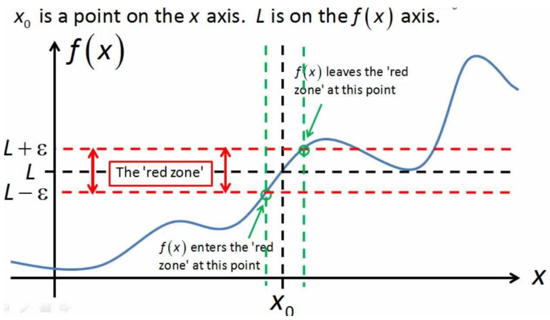
   Some graphics illustration for limit epsilon definitions :



Given a yellow "flat tube" as small as you want centered at L you will always find a blue "flat tube" centered at x_0 such that the images of the points on the blue flat tube end up on the yellow one. (as the yellow flat tube becomes small the blue flat tube becomes small, meaning you come closer to L as you come closer to x_0. Exactly the definition of the limit.)

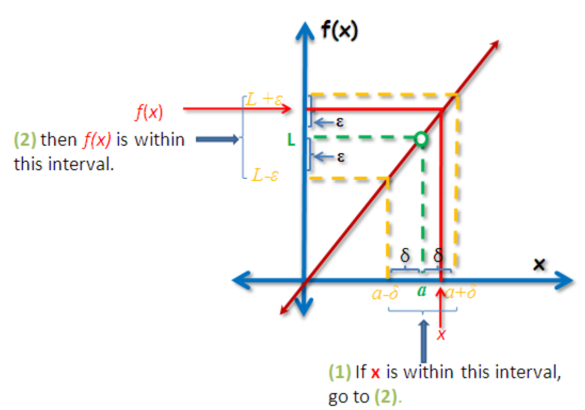
Given a red "flat tube" as small as you want centered at L you will always find a purple "flat tube" centered at x_0 such that the images of the points on the blue flat tube end up on the red one.





Given a red "flat tube" as small as you want centered at L you will always find a green "flat tube" around x_0 such that the images of the points on the green flat tube end up on the red one.

Here an example to make sure you understood perfectly what is going on.



How you can think about it is I give you a tube 1 whose diameter is 2ϵ, you put this tube centered on L on your y-axis then you try to construct a tube 2 with a certain diameter that you can choose as small as you want that you put centered on a, if you can find a tube small enough such that all the possible images for points of tube 2 ends up in tube 1. It means that you can get at a distance epsilon to L if you stay at a certain distance from a. Then I give you a small tube (smaller epsilon), and if you can do it again for any tube I give you as small as I want then it means you manage to get as close to L as I wanted to be by staying close enough to a. Here an example where the limit is not L Try to find a tube around a such that all the image of the point in the tube say in $(-\epsilon + L, \epsilon + L)$, I believe you wont manage, because the limit is not L at a... I think we are ready now for serious staff, how do I write this formally, in a proper way,

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \textit{such that if}\ 0 < |x - a| < \delta\ \textit{then}\ |f(x) - f(a)| < \epsilon$$

What this mathematical language means : for any $\epsilon > 0$ (distance as small as I want to choose it, this is tube 1 diameter I give you), if you are able find a $\delta > 0$ distance that you can choose as small as you need to (diameter for tube 2) such that for any point x in this tube, $f(x)$ is in tube 1, meaning $|f(x) - L| < \epsilon$. I will give you the other mathematical definition without explanation, you have to see if you can figure out what they mean.

2. $lim_{x \to a^+} f(x) = L$

$$\forall \epsilon > 0, \exists \delta > 0,\ \textit{such that if}\ a < x < a + \delta\ \textit{then}\ |f(x) - L| < \epsilon$$

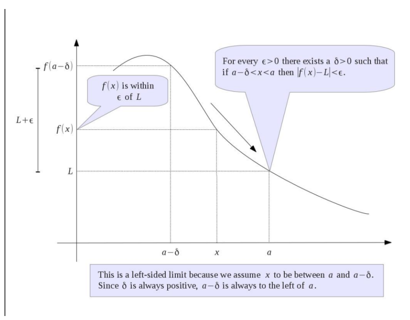
English translation : I can give you any tube as small as I want say with diameter ϵ that I center at L, you should be able to find $\delta > 0$ as small as you need to so that for any $a < x < a + \delta$ (tube from a to $a + \epsilon$ tube on the right of a), the image of those x end up all in the tube I created first, that is $|f(x) - L| < \epsilon$.

3. $lim_{x \to a^-} f(x) = L$

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that if } a - \delta < x < a \text{ then } |f(x) - L| < \epsilon$$

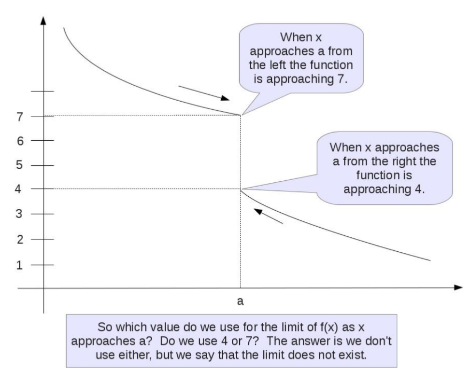
English translation : I can give you any tube as small as I want say with diameter ϵ that I center at L, you should be able to find $\delta > 0$ as small as you need to so that for any $a - \delta < x < a$ (tube from $a - \delta$ to a tube on the left of a), the image of those x end up all in the tube I created first, that is $|f(x) - L| < \epsilon$.

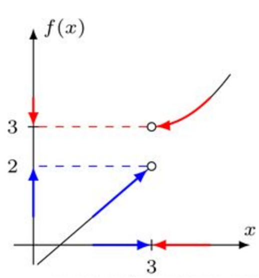
Some graphic illustrations for those limit epsilon definitions :



Given a "flat tube" as small as you want centered at L on the y-axis you will always find a "flat tube" on the left of a touching a such that the images of the points on the latter flat tube end up in the first one.

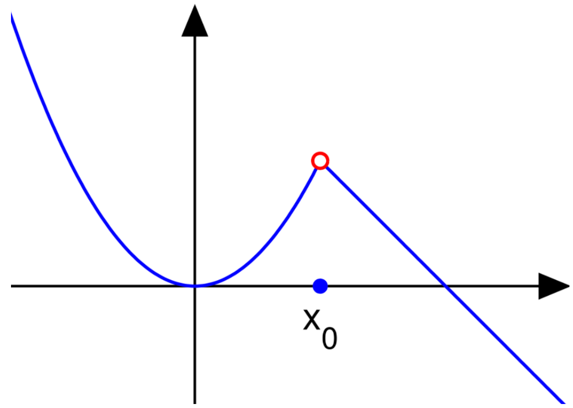
Here you can see what is the limit on the right and on the left, they both exist but do not coincide thus you do not have a limit at a only at a^+ and a^-.

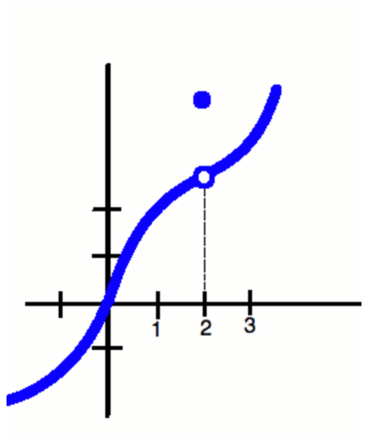




Here you can see what is the limit on the right and on the left, they both exist but do not coincide thus you do not have a limit at a only at a^+ and a^-.

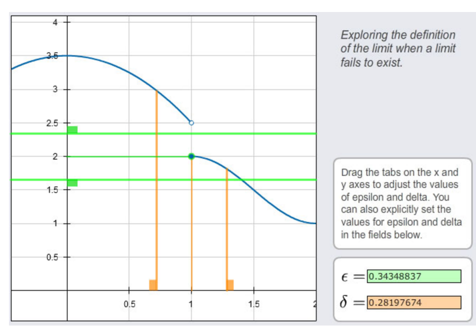
Here you can see what is the limit on the right and on the left, they both exist and coincide thus the limit at x_0 is this common limit even if the point x_0 is not part of the domain of definition of f.





Here you can see what is the limit on the right and on the left at 2, they both exist and coincide thus the limit at 2 is this common limit even if at the point 2 the image is something else

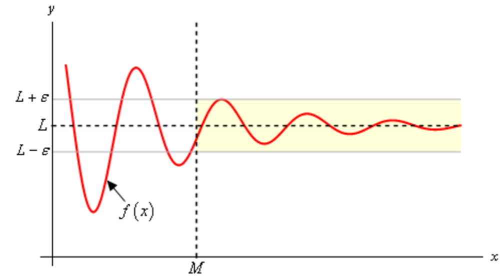
Here you can see what is the limit on the right and on the left of 1, they both exist but do not coincide thus you do not have a limit at 1 only at 1^+ and 1^-.



4. $lim_{x\to\infty} f(x) = L$ where $L \in \mathbb{R}$.

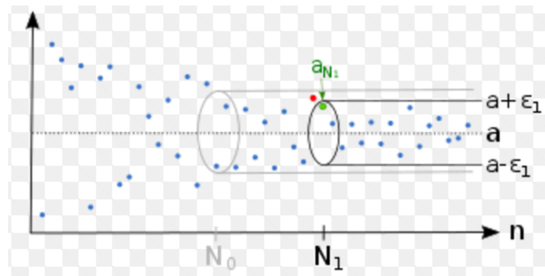
$$\forall\epsilon > 0, \exists M > 0, \text{ such that if } x > M \text{ then } |f(x) - L| < \epsilon$$

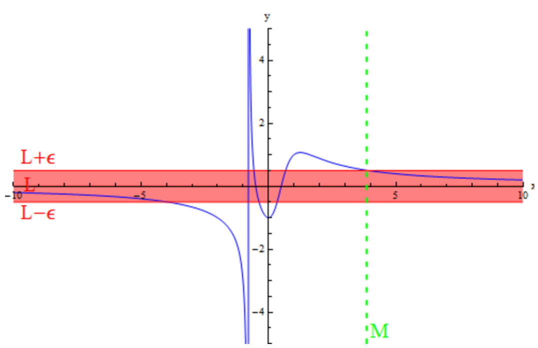
English translation : I can give you any tube as small as I want say with diameter 2ϵ that I center at L, you should be able to find $M > 0$ big such that for any $x > M$, the image of those x end up all in the tube I created first, that is $|f(x) - L| < \epsilon$.
Some graphic illustrations for those limit epsilon definitions :



Here if give you a yellow tube centered at L, you can give me $M > 0$ (as in the picture, for instance) such that every single x bigger than this M has an image falling in the yellow tube I first gave you, this means as x goes bigger $f(x)$ goes closer to L.

Here, I am working with sequences for a change : if give you the tube centered at a, you can give me $N_0 > 0$ (as in the picture, for instance) such that every single n bigger than this N_0 has an image falling in the tube I first gave you, if I give you a smaller tube, you will still be able to find me a N_1 (as in the picture, for instance) such that every single n bigger than this N_1 has an image falling in this smaller tube, this means as n goes bigger $f(n)$ goes closer to a.





Here if give you a red flat tube centered at L, you can give me $M > 0$ (as in the picture, for instance) such that every single x bigger than this M has an image falling in the red flat tube I first gave you, this means as x goes bigger $f(x)$ goes closer to L.

5. $lim_{x \to -\infty} f(x) = L$ where $L \in \mathbb{R}$.

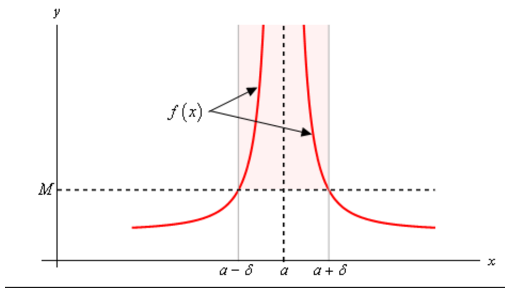
$$\forall \epsilon > 0, \exists M < 0, \ such \ that \ if \ x < M \ then \ |f(x) - L| < \epsilon$$

   English translation : I can give you any tube as small as I want say with diameter 2ϵ that I center at L, you should be able to find $M < 0$ big in the negative such that for any $x < M$, the images $f(x)$ of those x end up all in the tube I created first, that is $|f(x) - L| < \epsilon$.

6. $lim_{x \to a} f(x) = \infty$ where $a \in \mathbb{R}$.

$$\forall M > 0, \exists \delta > 0, \ such \ that \ if \ 0 < |x - a| < \delta \ then \ f(x) > M$$

   English translation : I can give you any number $M > 0$ as big as I want to, you should be able to find a tube as small as you want that you center at a such that the images $f(x)$ for any point x in this tube end up being greater than M, that is $f(x) > M$.



Here if give you $M > 0$, you can find for me a pink flat tube centered on a such that every single x in this tube has an image bigger than the M I gave you.

7. $lim_{x \to a^+} f(x) = \infty$ where $a \in \mathbb{R}$.

$$\forall M > 0, \exists \delta > 0, \text{ such that if } a < x < \delta + a \text{ then } f(x) > M$$

   English translation : I can give you any number $M > 0$ as big as I want to, you should be able to find a tube from a to $a + \delta$ with $\delta > 0$ such that the images $f(x)$ for any point x in this tube end up being greater than M, that is $f(x) > M$.

8. $lim_{x \to a^-} f(x) = \infty$ where $a \in \mathbb{R}$.

$$\forall M > 0, \exists \delta > 0, \text{ such that if } a - \delta < x < a \text{ then } f(x) > M$$

   English translation : I can give you any number $M > 0$ as big as I want to, you should be able to find a tube from $a - \delta$ to a with $\delta > 0$ such that the images $f(x)$ for any point x in this tube end up being greater than M, that is $f(x) > M$.

9. $lim_{x \to a} f(x) = -\infty$ where $a \in \mathbb{R}$.

$$\forall M < 0, \exists \delta > 0, \text{ such that if } 0 < |x - a| < \delta \text{ then } f(x) > M$$

   English translation : I can give you any number $M < 0$ as big as I want to in the negative, you should be able to find a tube as small as you want that you center at a such that the images $f(x)$ for any point x in this tube end up being smaller than M, that is $f(x) < M$.

10. $lim_{x \to a^+} f(x) = -\infty$ where $a \in \mathbb{R}$.

$$\forall M < 0, \exists \delta > 0, \text{ such that if } a < x - < a + \delta \text{ then } f(x) > M$$

   English translation : I can give you any number $M < 0$ as big as I want to in the negative, you should be able to find a tube from a to $a + \delta$ for some $\delta > 0$ such that the images $f(x)$ for any point x in this tube end up being smaller than M, that is $f(x) < M$.

11. $lim_{x \to a^-} f(x) = -\infty$ where $a \in \mathbb{R}$.

$$\forall M < 0, \exists \delta > 0, \text{ such that if } a - \delta < x - < a \text{ then } f(x) > M$$

   English translation : I can give you any number $M < 0$ as big as I want to in the negative, you should be able to find a tube from $a - \delta$ to a for some $\delta > 0$ such that the images $f(x)$ for any point x in this tube end up being smaller than M, that is $f(x) < M$.

12. $lim_{x\to\infty}f(x) = \infty$.

$$\forall M > 0, \exists N > 0, \textit{ such that if } x > N \textit{ then } f(x) > M$$

English translation : I can give you any number $M > 0$ as big as I want to, you should be able to find a big number $N > 0$, such that the images of any x bigger than this N $(x > N)$ are greater than M $(f(x) > M)$.

13. $lim_{x\to\infty}f(x) = -\infty$.

$$\forall M < 0, \exists N > 0, \textit{ such that if } x > N \textit{ then } f(x) < M$$

English translation : I can give you any number $M < 0$ as big as I want to in the negative, you should be able to find a big number $N > 0$, such that the images of any x bigger than this N $(x > N)$ are smaller than M $(f(x) < M)$.

14. $lim_{x\to-\infty}f(x) = \infty$.

$$\forall M > 0, \exists N < 0, \textit{ such that if } x < N \textit{ then } f(x) > M$$

English translation : I can give you any number $M > 0$ as big as I want to, you should be able to find a big number $N < 0$ in the negative, such that the images of any x smaller than this N $(x < N)$ are bigger than M $(f(x) > M)$.

15. $lim_{x\to-\infty}f(x) = -\infty$.

$$\forall M < 0, \exists N < 0, \textit{ such that if } x < N \textit{ then } f(x) < M$$

English translation : I can give you any number $M < 0$ as big as I want to in the negative, you should be able to find a big number $N < 0$ in the negative, such that the images of any x smaller than this N $(x > N)$ are smaller than M $(f(x) < M)$.

Formal definition are here to make proof, the results you accept to be true are true because you can prove them, and those definitions sometimes help you to prove some limits results.

Somethings you can learn from those formal definitions : when you are looking at a limit $lim_{x\to a}f(x) = L$, the only thing that matters is what is happening around a, so you do not need to know your function too far away from the point a ; if $a \in \mathbb{R}$, you can zoom in around your a with intervals of the form $(a - \delta, a + \delta)$ where $\delta > 0$, when $a = \infty$, then you can zoom in around a with intervals of the form (M, ∞) where $M > 0$ and when $a = -\infty$, then you can zoom in around a with intervals of the form $(-\infty, M)$ where $M < 0$.

Continuity

1. We say that f **is continuous at** a that belongs to the domain of definition of f, if

$$lim_{x \to a} f(x) = f(a).$$

   In any other case, we say that f is **discontinuous at** a.

2. We say that f **is continuous over an interval** I if f is continuous at any point of the interval I.

The following function are continuous over their DOMAIN OF DEFINITION :

1. polynomials ;
2. rational functions ;
3. root functions ;
4. trigonometric functions ;
5. exponential functions ;
6. logarithmic functions ;

Moreover, if f and g are continuous at a, then the following function are continuous at a :

1. $f + g$;
2. $f - g$;
3. fg ;
4. $\frac{f}{g}$, when $g(a) \neq 0$.

and any combination of those.

Helpful limits

$$lim_{x\to\infty}x^n = \infty;$$

$$lim_{x\to-\infty}x^n = \infty, \text{ when } n \text{ is even natural number, and}$$

$$lim_{x\to-\infty}x^n = -\infty, \text{ when } n \text{ is odd natural number.}$$

$$lim_{x\to\infty}\frac{1}{x^n} = 0;$$

$$lim_{x\to-\infty}\frac{1}{x^n} = 0;$$

$$lim_{x\to 0^+}\frac{1}{x^n} = \infty;$$

$$lim_{x\to 0^-}\frac{1}{x^n} = \infty, \text{ when } n \text{ is even natural number, and}$$

$$lim_{x\to 0^-}\frac{1}{x^n} = -\infty, \text{ when } n \text{ is odd natural number.}$$

$$lim_{x\to\infty}e^x = \infty;$$

$$lim_{x\to-\infty}e^x = 0;$$

$$lim_{x\to\infty}ln(x) = \infty;$$

$$lim_{x\to 0^+}ln(x) = -\infty;$$

cos and sin have no limit at both, ∞ and $-\infty$.

Composition rules

Theorem 0.3. *If* $lim_{x\to a}g(x) = A$ *and* $lim_{X\to A}f(X) = L$ *then*

$$lim_{x\to a}f(g(x)) = L$$

Limits rule tables

In the following we suppose a to be either a real number or ∞ or $-\infty$.

$lim_{x\to a}f(x)$	$lim_{x\to a}g(x)$	$lim_{x\to a}(f(x)+g(x))$	$lim_{x\to a}(f(x)-g(x))$
$L\in\mathbb{R}$	$L'\in\mathbb{R}$	L+ L'	L-L'
$L\in\mathbb{R}$	∞	∞	$-\infty$
$L\in\mathbb{R}$	$-\infty$	$-\infty$	∞
∞	$L\in\mathbb{R}$	∞	∞
$-\infty$	$L\in\mathbb{R}$	$-\infty$	$-\infty$
∞	∞	∞	INDETERMINATE
∞	$-\infty$	INDETERMINATE	∞
$-\infty$	∞	INDETERMINATE	$-\infty$
$-\infty$	$-\infty$	$-\infty$	INDETERMINATE

$lim_{x\to a}f(x)$	$lim_{x\to a}g(x)$	$lim_{x\to a}(f(x)g(x))$
$L\in\mathbb{R}$	$L'\in\mathbb{R}$	L L'
$L>0$	∞	∞
$L<0$	∞	$-\infty$
$L>0$	$-\infty$	$-\infty$
$L<0$	$-\infty$	∞
$L=0$	∞	INDETERMINATE
$L=0$	$-\infty$	INDETERMINATE

$lim_{x\to a}f(x)$	$lim_{x\to a}g(x)$	$lim_{x\to a}\frac{f(x)}{g(x)}$
$L\in\mathbb{R}$	$L'\in\mathbb{R},\ L'\neq 0$	$\frac{L}{L'}$
$L>0$	$L'=0^+$	∞
$L>0$	$L'=0^-$	$-\infty$
$L<0$	$L'=0^+$	$-\infty$
$L<0$	$L'=0^-$	∞
$L=0$	$L'=0$	INDETERMINATE
$L\in\mathbb{R}$	∞	0
$L\in\mathbb{R}$	$-\infty$	0
∞	$L'=0^+$	∞
∞	$L'=0^-$	$-\infty$
$-\infty$	$L'=0^+$	$-\infty$
$-\infty$	$L'=0^-$	∞
∞	$L'>0$	∞
∞	$L'<0$	$-\infty$
$-\infty$	$L'>0$	$-\infty$
$-\infty$	$L'<0$	∞
∞	∞	INDETERMINATE
∞	$-\infty$	INDETERMINATE
$-\infty$	∞	INDETERMINATE
$-\infty$	$-\infty$	INDETERMINATE

Some tricks to answer limits problems

Step for a limit problem $lim_{x \to a} f(x) = L$ **you should do in your draft/head before answering a question :**

1. First question to ask yourself could be is the function defined close to the point a because if it is not defined around the point a then there is not even a function to look at thus the limit does not even begin to exist. You need to be able to take value close to your point (if you are looking at limits such as $lim_{x \to a^+} f(x) = L$ you need to be able to take value for your function on the right and if you are looking at limits such as $lim_{x \to a^+} f(x) = L$ you need to be able to take value for your function on the left.)
2. If a is a real number, THINK is a in the domain of definition of the function where f is continuous. If so you do not have much to do since
$$lim_{x \to a} f(x) = f(a)$$
and you only need to compute $f(a)$.
3. If you are not in the previous case, SEE FIRST if just trying to find out the limit the most obvious way do not work before doing any transformation. LOOK AT THE ABOVE TABLE TO SEE IF THE FORM YOU GET IS OR NOT AN INDETERMINATE. Maybe all you will need ONLY to separate left and right limit to get an answer !!!!
4. If the form is an **indeterminate form** : IT DOES NOT MEAN YOU CANNOT GET AN ANSWER : IT MEANS YOU NEED TO FIND A WAY TO REWRITE YOUR FUNCTION OR USE SOME THEOREM SEEN IN CLASS (Squeeze theorem...) **TO ANSWER THE QUESTION. DO NOT DO THIS IF YOU DO NOT NEED TO FIRST CHECK STEP 1. AND 2. IN YOUR HEAD OR DRAFT.**

Do not forget I cannot give you enough tricks to solve all the problems of the world so sometimes you mind have to find a unique idea on your own to solve a problem. Here some tips (SOMETIMES YOU MIGHT NEED TO COMBINE SEVERAL OF THOSE TO COMPUTE ONE LIMIT) :

1. If I am asking you the limit of a polynomial at ∞ or $-\infty$, it is an easy question just put in factor the higher power of your indeterminate if necessary !!!!
   (a) $lim_{x \to \infty} x^3 - 3x + 3$
   $$x^3 - 3x + 3 = x^3(1 - \frac{3}{x^2} + \frac{3}{x^3})$$
   We know that
   $$lim_{x \to \infty} 1 - \frac{3}{x^2} + \frac{3}{x^3} = 1 - 0 + 0$$
   and
   $$lim_{x \to \infty} x^3 = \infty$$
   Thus
   $$lim_{x \to \infty} x^3 - 3x + 3 = \infty$$

(b) $lim_{x\to\infty}x^3 + 3x$

You can use the same method as before it will work very well or also do the limit directly as we have not indeed an indeterminate :

$$lim_{x\to\infty}x^3 = \infty$$

and

$$lim_{x\to\infty}3x = \infty$$

thus

$$lim_{x\to\infty}x^3 + 3x = \infty$$

(c) $lim_{x\to-\infty}x^2 + 3$

Again you can use the same method as in (a) but you do not really need that indeed here it is easy as

$$lim_{x\to-\infty}x^2 = \infty$$

thus

$$lim_{x\to-\infty}x^2 + 3 = \infty$$

(d) $lim_{x\to 3}x^2 - 7$

DO NOT DO UNNECESSESARY WORK : I am asking a limit at 3 real number not infinity so the limit is just the value of the polynomial at 3, Be clever, do not jump and think of the steps explained before.

$$lim_{x\to 3}x^2 - 7 = 2$$

2. In order to compute the limit of a rational function at ∞ or $-\infty$ **(not at a real number)** you can put the higher power of the indeterminate of the numerator in factor in the numerator and the highest power of the indeterminate of the denominator in factor in the denominator.

(a) $lim_{x\to\infty}\frac{x^3+3x}{x^2+3x+2}$ First, work without the limit symbol you will avoid a bit of the mess on your test

$$\frac{x^3 + 3x}{x^2 + 3x + 2} = \frac{x^3(1 + \frac{3}{x^2})}{x^2(1 + \frac{3}{x} + \frac{2}{x^2})} = x\frac{(1 + \frac{3}{x^2})}{(1 + \frac{3}{x} + \frac{2}{x^2})}$$

We know that

$$lim_{x\to\infty}\frac{(1 + \frac{3}{x^2})}{(1 + \frac{3}{x} + \frac{2}{x^2})} = \frac{1+0}{1+0+0} = 1$$

and

$$lim_{x\to\infty}x = \infty$$

Using the tables, we get

$$lim_{x\to\infty}\frac{x^3 + 3x}{x^2 + 3x + 2} = \infty$$

(b) $lim_{x \to -\infty} \frac{x^3+3x}{x}$

We could do the same method as before and it would work but let's not be a robot and think a bit before jumping the guns :

You can see that

$$\frac{x^3 + 3x}{x} = x^2 + 3$$

and since $lim_{x \to -\infty} x^2 = \infty$ thus

$$lim_{x \to -\infty} \frac{x^3 + 3x}{x} = lim_{x \to -\infty} x^2 + 3 = \infty$$

3. In order to compute the limit of a rational function at a real number, first check if you get some kind of indeterminate if the real number is in the domain of definition of your function then it is all fine your function is continuous you just need to evaluate the function at this number, if the real number is not in the domain of definition of your function then there is two options either the tables above give you an answer you might have to separate right and left limit to answer or you have an indeterminate form like 0 over 0, in this case you should be able to factor something on the top and bottom and simplify to get your answer.

(a) $lim_{x \to 1} \frac{x^2+3}{x^3+x}$

Let's not go too fast look the denominator is not 0 so 1 is part of the domain of the definition so it is not really a complicated question I am asking you. How to answer :

Since 1 is part of the domain of the definition of this rational function which is a continuous function we have by definition of continuity,

$$lim_{x \to 1} \frac{x^2 + 3}{x^3 + x} = \frac{1^2 + 3}{1^3 + 1} = 2$$

(b) Compute $lim_{x \to 1^+} \frac{x^2+3}{x^3-x}$, $lim_{x \to 1^-} \frac{x^2+3}{x^3-x}$ and $lim_{x \to 1} \frac{x^2+3}{x^3-x}$.

Now it is a different story, you see that the denominator is 0 when you plug 1 but the numerator is not 0 so there is no problem here we can decide what is going on without much work. Here how to answer :

First, note that

$$lim_{x \to 1} x^2 + 3 = 4 > 0$$

and that $x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)$ and that when $x > 1$, then $x(x + 1)(x - 1) > 0$ and when $0 < x < 1$, $x(x + 1)(x - 1) < 0$. Thus,

$$lim_{x \to 1^+} x^3 - x = 0^+$$

and using the table

$$lim_{x \to 1^+} \frac{x^2 + 3}{x^3 - x} = \infty$$

Also

$$lim_{x \to 1^-} x^3 - x = 0^-$$

and using the table

$$lim_{x \to 1^-} \frac{x^2+3}{x^3-x} = -\infty$$

Since the limit on the right and the left do not coincide, we know that $lim_{x \to 1} \frac{x^2+3}{x^3-x}$ does not exist.

(c)

$$lim_{x \to 1} \frac{x-1}{x^3-x}$$

Here, you can quickly notice that if you plug one the numerator and denominator is zero, well looking at the table you can see that this is an undeterminate form. What can we do ? Here is how you can answer :

Note that $x^3 - x = x(x-1)(x+1)$ thus

$$\frac{x-1}{x^3-x} = \frac{1}{(x+1)x}$$

and now the right side of the equality is perfectly defined and continuous at 1 thus

$$lim_{x \to 1} \frac{x-1}{x^3-x} = lim_{x \to 1} \frac{1}{(x+1)x} = \frac{1}{2}$$

4. For limit involving absolute value or piecewise function, you need to figure out what is your function close to the point you are considering. You might need to separate left and right limit to get an answer. Remember, being close ∞ is being in an interval of the form (M, ∞) with $M > 0$ big, being close to $-\infty$ is being in an interval of the form $(-\infty, M)$ with $M < 0$ and $-M$ big, being close to a real number is being on an interval of the form $(a-\delta, a+\delta)$ where $\delta > 0$ small, being close to a on the right is being on an interval of the form $(a, a+\delta)$ where $\delta > 0$ small and being close to a on the left is being on an interval of the form $(a-\delta, a)$ where $\delta > 0$ small.

   (a) $lim_{x \to -3} |x+1| + \frac{3}{x}$

   How you should think in your draft/head : Well if I plug -3 there is not really nothing to worry about the function is defined are continuous at -3. So that is an easy question.

   How you can answer :

   Since the function defined by $|x+1| + \frac{3}{x}$ and continuous at -3 we have

$$lim_{x \to -3} |x+1| + \frac{3}{x} = |-3+1| + \frac{3}{-3} = 2 - 1 = 1$$

   (b) $lim_{x \to 1} \frac{x^2-1}{|x-1|}$

   How you should think in your draft/head : I have an absolute value how can I do. Well lets try first to do a simple replacement and see what happens. Well the denominator and numerator are 0 so indeterminate I need to simplify. Well with the absolute value of $|x-1|$ I see that this expression depends on $x > 1$ or $x < 1$ so I will need to separate the limit on the right and left if I want to get somewhere.

How you can answer : When $x > 1$

$$\frac{x^2-1}{|x-1|} = \frac{(x-1)(x+1)}{x-1} = x+1$$

Thus

$$lim_{x\to 1^+}\frac{x^2-1}{|x-1|} = lim_{x\to 1^+} x+1 = 2$$

When $x < 1$,

$$\frac{x^2-1}{|x-1|} = \frac{(x-1)(x+1)}{-(x-1)} = -(x+1)$$

Thus

$$lim_{x\to 1^-}\frac{x^2-1}{|x-1|} = lim_{x\to 1^-} -(x+1) = -2$$

We see that the limit on the right an left do not coincide thus $lim_{x\to 1}\frac{x^2-1}{|x-1|}$ does not exist.

(c) $lim_{x\to -2}\frac{1}{|x+2|} + x^2$

How you should think in your draft/head : Let first see as usual what happens when I plug -2, well x^2 no problem for $\frac{1}{|x+2|}$ the numerator 1 and the denominator is 0 but positive since the absolute value is always positive cool I know this limit with the quotient limit table.

How you can answer :

$$lim_{x\to -2} x^2 = (-2)^2 = 4$$

and

$$lim_{x\to -2}|x+2| = 0^+$$

thus using the quotient limit table we get

$$lim_{x\to -2}\frac{1}{|x+2|} = \infty$$

and using the sum limit table we get

$$lim_{x\to -2}\frac{1}{|x+2|} + x^2 = \infty$$

(d) $lim_{x\to 3^-}\frac{x^2|x-3|}{x-3}$

How you should think in your draft/head : Let first see as usual what happens when I plug 3, well I get 0 in numerator and denominator, indeterminate grrrr. Well it is not so bad as when you close to 3 on the left (meaning $x < 3$), $|x-3| = -(x-3)$, I should be able to simplify then.

How you can answer : When $x < 3$,

$$\frac{x^2|x-3|}{x-3} = \frac{-x^2(x-3)}{x-3} = -x^2$$

thus

$$lim_{x\to 3^-}\frac{x^2|x-3|}{x-3} = lim_{x\to 3^-} -x^2 = -3^2 = -9$$

5. Sometimes when square root are involved and YOU GET AN INDETERMINATE FORM the way to go is to rationalize. But do not jump and rationalize sometimes you do not need to. Observe do the steps describe above and if you get an indeterminate try out a rationalization :

   (a) $lim_{x\to 3}\frac{\sqrt{x+1}-2}{x^2-9}$

   How you should think in your draft/head : Well if I plug 3 we get zero for denominator and numerator so I got indeterminate form. Since there is a square root maybe rationalizing help me to get rid of the square root.

   How you can answer :

$$\begin{aligned} \frac{\sqrt{x+1}-2}{x^2-9} &= \frac{\sqrt{x+1}-2}{x^2-9}\times\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \\ &= \frac{x-3}{(x-3)(x+3)(\sqrt{x+1}+2)} = \frac{1}{(x+3)(\sqrt{x+1}+2)} \end{aligned}$$

   Since the function defined by $\frac{1}{(x+3)(\sqrt{x+1}+2)}$ is continuous at 3 we have

$$lim_{x\to 3}\frac{\sqrt{x+1}-2}{x^2-9} = lim_{x\to 3}\frac{1}{(x+3)(\sqrt{x+1}+2)} = \frac{1}{24}$$

   (b) $lim_{x\to 3}\frac{\sqrt{x^2+7}-3}{x+3}$

   How you should think in your draft/head : Well if I plug 3 there is absolutely no problem the denominator is not 0. Great, it is a super easy question.

   How you can answer : The function defined by $\frac{\sqrt{x^2+7}-3}{x+3}$ is well defined and continuous at 3 thus

$$lim_{x\to 3}\frac{\sqrt{x^2+7}-3}{x+3} = \frac{\sqrt{3^2+7}-3}{3+3} = \frac{8}{3}$$

6. Trigonometric function as cos and sin have no limit at infinity. But SOMETIMES, we can still make use of the squeeze theorem when they are involved and get a limit. Again, do not JUMP each time you have a cosinus or sinus the Squeeze theorem is not always the answer.

   (a) $lim_{x\to 0}\sqrt{2cos(x)-5}$

   How you should think in your draft/head : Well no where near 0 this function is define we are taking the square root of a negative number even if I am very near 0.

   How you can answer :

   We know that for any $x\in\mathbb{R}$, $-1\leqslant cos(x)\leqslant 1$ thus $-7\leqslant 2cos(x)-5\leqslant -3$. So the function defined by $\sqrt{2cos(x)-5}$ is never defined over the real number and since we cannot speak about the function we cannot even speak about any limit.

   (b) $lim_{x\to\infty}\frac{cos(x)}{e^x}$

   How you should think in your draft/head : Here, I cannot really answer to this directly, I have a cos so maybe the squeeze theorem can help me.

   How you can answer :

   We know that

$$-1\leqslant cos(x)\leqslant 1$$

for any real number x. Thus

$$-1/e^x \leqslant cos(x)/e^x \leqslant 1/e^x$$

We know that

$$lim_{x\to\infty}e^x = \infty$$

thus

$$lim_{x\to\infty} - 1/e^x = lim_{x\to\infty}1/e^x = 0$$

You can now apply the Squeeze theorem and you get

$$lim_{x\to\infty}cos(x)/e^x = 0$$

BE AWARE, ONLY WHEN THE LIMIT IN THE RIGHT AND LEFT OF INEQUALITIES COINCIDE YOU CAN CONCLUDE USING THE SQUEEZE THEOREM

(c) $lim_{x\to 0}(x^2+1)sin(\frac{1}{x})$

How you should think in your draft/head : Here, if you think a bit it seems that the squeeze theorem might be necessary but you get $-(x^2+1) \leqslant (x^2+1)sin(\frac{1}{x}) \leqslant x^2+1$

$$lim_{x\to 0}x^2 + 1 \neq lim_{x\to 0} - (x^2+1)$$

THE SQUEEZE THEOREM DO NOT APPLY DIRECTLY. Don't stress we might have to do a bit more work.

How you can answer :

Note that

$$(x^2+1)sin(\frac{1}{x}) = x^2 sin(\frac{1}{x}) + sin(\frac{1}{x})$$

We have

$$-1 \leqslant sin(\frac{1}{x}) \leqslant 1$$

thus

$$-x^2 \leqslant x^2 sin(\frac{1}{x}) \leqslant x^2$$

and

$$lim_{x\to 0}x^2 = lim_{x\to 0} - x^2 = 0$$

Thus by the squeeze theorem,

$$lim_{x\to 0}x^2 sin(\frac{1}{x}) = 0$$

Now, $lim_{x\to 0^+}\frac{1}{x} = \infty$ and $lim_{X\to\infty}sin(X)$ does not exist thus $lim_{x\to 0^+}sin(\frac{1}{x})$ does not exist.

We can already conclude that $lim_{x\to 0}(x^2+1)sin(\frac{1}{x})$

(d) $lim_{x\to\infty}\frac{sin(\frac{1}{x})}{x^2}$

How you should think in your draft/head : Let's not be naive and not each time we see a cos or sin think that the squeeze theorem is the solution. See this example, $lim_{x\to\infty}frac1x = 0$ thus $lim_{x\to\infty}sin(frac1x) = 0$ and $lim_{x\to\infty}frac1x^2 = 0$, so I DO NOT NEED THE SQUEEZE THEOREM HERE.

How you can answer :

$lim_{x\to\infty}frac1x = 0$ and $lim_{X\to 0}sin(X) = 0$, thus using the composite rule, $lim_{x\to\infty}sin(frac1x) = 0$. Moreover,

$$lim_{x\to\infty}frac1x^2 = 0$$

Thus using the product table

$$lim_{x\to\infty}\frac{sin(\frac{1}{x})}{x^2} = lim_{x\to\infty}\frac{1}{x^2}sin(\frac{1}{x}) = 0$$

7. When you have composite function you can use the composition rule.

(a) $lim_{x\to\infty}\sqrt{x^2-2} - \sqrt{x^2+1}$

How you should think in your draft/head : Quickly, you can intuitively think at infinity $\sqrt{x^2-2}$ should be ∞ and $-\sqrt{x^2+1}$ is $-\infty$. This is an indeterminate for sums. Since we have a square root MAYBE rationalizing could help lets try.

How you can answer :

We have

$$\begin{aligned}\sqrt{x^2-2} - \sqrt{x^2+1} &= \frac{(\sqrt{x^2-2} - \sqrt{x^2+1})(\sqrt{x^2-2} + \sqrt{x^2+1})}{(\sqrt{x^2-2} + \sqrt{x^2+1})} \\ &= \frac{-3}{(\sqrt{x^2-2} + \sqrt{x^2+1})}\end{aligned}$$

Since $lim_{x\to\infty}x^2 - 2 = lim_{x\to\infty}x^2 + 1 = \infty$ then using the composition rule since $lim_{X\to\infty}\sqrt{X} = \infty$, $lim_{x\to\infty}\sqrt{x^2-2} = lim_{x\to\infty}\sqrt{x^2+1} = \infty$. Thus, using the table limits we get

$$lim_{x\to\infty}\sqrt{x^2-2} - \sqrt{x^2+1} = lim_{x\to\infty}\frac{-3}{(\sqrt{x^2-2} + \sqrt{x^2+1})} = 0$$

You can find plenty of very detailed examples in tutorial 5.

- **WELL, IF YOU UNDERSTOOD ME, you can see I give you plenty of trick but they will never be enough to answer all the limit question of the world. You always need to think of the best method which applies. It is not because you have a rational function you have to rationalize, it is not because you have a trigonometric function that you have to use the squeeze theorem it is not because you have a rational function that you need to factorize. You need to be clever and not a robot. So, THINK BEFORE JUMPING INTO THE ANSWER AND THEN TAKE THE PLUNGE.**
- **BE AWARE : That is you get, something that is not 0 ($-\infty$, ∞, $c \neq 0$ real number) over 0 it is NOT a indeterminate form and you need to find the sign of 0. In order to determine it you can use (sign table for instance). Note : you only need to know the sign of the denominator close to a whatever a is. For instance, if a is infinity you only need to find the sign when of $f(x)$ when x is large....**

Hospital rule

Theorem 0.4 (Hospital rule)**.** *Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that*

$$lim_{x \to a} f(x) = 0 \text{ and } lim_{x \to a} g(x) = 0$$

or that

$$lim_{x \to a} f(x) = \pm\infty \text{ and } lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$lim_{x \to a} \frac{f(x)}{g(x)} = lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

When using the hospital rule you need to make sure to let me know you are using it. Before this you need to state that all the conditions needed to apply the Hospital rule are there (differentiability, indeterminate form, derivatives...).

Also Hospital rules is not always the answer and if you find an straightforward algebraic way, a more basic method to find the limit it might be a better idea. Hospital rule is not always the best method.
The Hospital rule COULD maybe apply in one of the following case, keep this in mind :

- **Products**

$$fg = \frac{f}{1/g} = \frac{g}{1/f}$$

  The indeterminate for the product of the form $0 \cdot \infty$ can be converted into indeterminate of the from $\frac{0}{0}$ or ∞/∞ so we might be able to use the Hospital Rule.

- **Differences**

$$f - g$$

  The indeterminate for the difference (sum) of the form $\infty - \infty$ can sometimes be converted into quotient (for instance using common denominator, rationalizing, factoring...), so you might be able to use the Hospital Rule.

- **Powers Several indeterminate forms** arise from the limit

$$lim_{x \to a} [f(x)]^{g(x)}$$

  1. $lim_{x \to a} f(x) = 0$ and $lim_{x \to a} g(x) = 0$ type 0^0
  2. $lim_{x \to a} f(x) = \infty$ and $lim_{x \to a} g(x) = 0$ type ∞^0
  3. $lim_{x \to a} f(x) = 1$ and $lim_{x \to a} g(x) = \pm\infty$ type ∞^0

  Each of these case can be treated by taking the natural logarithm

$$y = [f(x)]^{g(x)} \text{ then } ln(y) = g(x) ln(f(x))$$

  or by writing the function as an exponential :

$$[f(x)]^{g(x)} = e^{g(x) ln(f(x))}$$

  And we get in each case 1., 2., 3. an indeterminate product $0 \cdot \infty$.

Here some limits also sometimes Hospital rule seems to be the answer but it is not.

1. $lim_{x \to -\infty} x^2 e^x$

   Solution : We write

   $$x^2 e^x = \frac{x^2}{e^{-x}}$$

   We can see that $f : x \mapsto x^2$ and $g : x \mapsto e^{-x}$ are differentiable over $\mathbb{R}$. And $lim_{x \to -\infty} x^2 = \infty$ and $lim_{x \to -\infty} - x = \infty$ and $lim_{X \to \infty} e^X = \infty$ therefore $lim_{x \to -\infty} e^{-x} = \infty$.

   We thus have a indeterminate form so all the condition required by the Hospital rule are satisfied and since $f'(x) = 2x$ and $g'(x) = -e^{-x}$. We get

   $$lim_{x \to -\infty} x^2 e^x = lim_{x \to -\infty} \frac{2x}{-e^{-x}}$$

   Again, we can see that $k : x \mapsto 2x$ and $l : x \mapsto -e^{-x}$ are differentiable over $\mathbb{R}$. And $lim_{x \to -\infty} 2x = -\infty$ and $lim_{x \to -\infty} - e^{-x} = -\infty$.

   We thus have a indeterminate form so all the condition required by the Hospital rule are satisfied and since $k'(x) = 2$ and $l'(x) = e^{-x}$. We get

   $$lim_{x \to -\infty} x^2 e^x = lim_{x \to -\infty} \frac{2x}{e^{-x}} = lim_{x \to -\infty} \frac{2}{e^{-x}} = lim_{x \to -\infty} 2e^x = 2 \times 0 = 0$$

2. $lim_{x \to -2} \frac{x^3 - x^2 - 10x - 8}{5x^3 + 12x^2 - 2x - 12}$;

   Solution : Note that if we plug -2 the numerator and denominator are 0 and we get indeterminate. Thus both polynomials in denominator and numerator are divisible by $x + 2$. Doing the division of those polynomial by $x + 2$ we get $x^3 - x^2 - 10x - 8 = (x - 4)(x + 2)(x + 1)$ and $5x^3 + 12x^2 - 2x = (x + 2)(5x^2 + 2x - 6)$.

   Then, for $x \neq -2$ we get

   $$\frac{x^3 - x^2 - 10x - 8}{5x^3 + 12x^2 - 2x - 12} = \frac{(x-4)(x+2)(x+1)}{(x+2)(5x^2 + 2x - 6)} = \frac{(x-4)(x+1)}{5x^2 + 2x - 6}$$

   Now the the right side is perfectly continuous at -2 and we get

   $$lim_{x \to -2} \frac{x^3 - x^2 - 10x - 8}{5x^3 + 12x^2 - 2x - 12} = lim_{x \to -2} \frac{(-2-4)(-2+1)}{5(-2)^2 + 2(-2) - 6} = \frac{6}{10} = \frac{3}{5}$$

3. $lim_{x \to \infty} \frac{tanh(x)}{tan^{-1}(x)}$;

   Here it is not really a big deal

   Solution : We have

   $$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^x(1 - e^{-2x})}{e^x(1 + e^{-2x})} = \frac{(1 - e^{-2x})}{(1 + e^{-2x})}.$$

   Moreover $lim_{x \to \infty} - 2x = -\infty$ and $lim_{X \to -\infty,\ X = -2x} e^X = 0$ thus by composition of limit rule we get

   $$lim_{x \to \infty} e^{-2x} = 0$$

   and thus

$$lim_{x\to\infty} tanh(x) = lim_{x\to\infty} \frac{(1-e^{-2x})}{(1+e^{-2x})} = \frac{1-0}{1+0} = 1$$

and $lim_{x\to\infty} tan^{-1}(x) = \pi/2$ since $lim_{\pi/2^-} tan(x) = \infty$.
Thus, by the limit table for quotient we get

$$lim_{x\to\infty} \frac{tanh(x)}{tan^{-1}(x)} = \frac{1}{\pi/2} = 2/\pi$$

4. $lim_{x\to 0} \frac{sin(x)}{sinh(x)}$;
   Here it seems that we might need the hospital rule
   Solution : The functions $h : x \mapsto sin(x)$ and $k : x \mapsto sinh(x)$ are differentiable at 0 and since both of those function are continuous at 0, we have $lim_{x\to 0} sin(x) = sin(0) = 0$ and $lim_{x\to 0} sinh(x) = sinh(0) = 0$.
   We notice that we have an undeterminate limit for the quotient 0 over 0. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $h'(x) = cos(x)$ and $k'(x) = cosh(x)$, by the Hospital rule we get

$$lim_{x\to 0} \frac{sin(x)}{sinh(x)} = lim_{x\to 0} \frac{cos(x)}{cosh(x)} = cos(0)/cosh(0) = 1$$

   The last limit is obtained using the continuity of $x \mapsto \frac{cos(x)}{cosh(x)}$.
5. $lim_{x\to 0} \left(\frac{e^x}{e^x-1} - \frac{1}{x}\right)$;
   Here it seems that Hospital rule might apply, but first we need to write the function as a quotient with indeterminate form of course !
   Solution :
   We have

$$\frac{e^x}{e^x-1} - \frac{1}{x} = \frac{xe^x - (e^x-1)}{x(e^x-1)} = \frac{xe^x - e^x + 1}{x(e^x-1)}$$

   The functions $h : x \mapsto xe^x - e^x + 1$ and $k : x \mapsto x(e^x - 1)$ are differentiable over $\mathbb{R}$ and since both of those function are continuous at 0, we have $lim_{x\to 0} xe^x - (e^x - 1) = 0e^0 - (e^0 - 1) = 0$ and $lim_{x\to 0} x(e^x - 1) = 0(e^0 - 1) = 0$.
   We notice that we have an undeterminate limit for the quotient 0 over 0. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $h'(x) = (x+1)e^x - e^x$ and $k'(x) = (e^x - 1) + xe^x$, by the Hospital rule we get

$$lim_{x\to 0} \frac{xe^x - e^x + 1}{x(e^x-1)} = lim_{x\to 0} \frac{xe^x}{e^x - 1 + xe^x}$$

   The functions $f : x \mapsto xe^x$ and $g : x \mapsto e^x - 1 + xe^x$ are differentiable at 0 and since both of those function are continuous at 0, we have $lim_{x\to 0} xe^x = 0e^0 = 0$ and $lim_{x\to 0} e^x - 1 + xe^x = e^0 - 1 + 0e^0 = 0$.
   We notice that we have an undeterminate limit for the quotient 0 over 0. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $h'(x) = (x+1)e^x$ and $k'(x) = (x+2)e^x$, by the Hospital rule we get

$$lim_{x\to 0} \frac{e^x}{e^x-1} - \frac{1}{x} = lim_{x\to 0} \frac{xe^x}{e^x - 1 + xe^x} = lim_{x\to 0} \frac{(x+1)e^x}{(x+2)e^x} = lim_{x\to 0} \frac{x+1}{x+2} = \frac{0+1}{0+2} = \frac{1}{2}$$

6. $lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2}$;
Solution : We know that the function $f : x \mapsto e^{x^2}$ is differentiable at 2 with derivative $f'(x) = 2xe^{x^2}$ once we apply the chain rule, by definition of the derivative at 2 of f we get

$$lim_{x\to 2}\frac{e^{x^2}-e^4}{x-2} = lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = f'(2) = 4e^4$$

7. $lim_{x\to 0}xsin(1/x)$;
Solution : We know that for any non zero real values, we have

$$-1 \leqslant sin(1/x) \leqslant 1$$

we have if $x > 0$

$$-x \leqslant xsin(1/x) \leqslant x$$

and if $x < 0$,

$$x \leqslant xsin(1/x) \leqslant -x$$

That is for any nonzero real values x

$$-|x| \leqslant xsin(1/x) \leqslant |x|$$

Moreover since the absolute value is continuous at 0, we have

$$lim_{x\to 0}|x| = lim_{x\to 0} - |x| = 0$$

Finally, by the squeeze theorem we have

$$lim_{x\to 0}xsin(1/x) = 0$$

8. $lim_{x\to \pi/2^-}\frac{sec(x)}{ln(sec(x))}$
Solution : We know that cos is continuous at $\pi/2^-$ thus

$$lim_{x\to \pi/2^-}cos(x) = 0^+$$

Thus

$$lim_{x\to \pi/2^-}sec(x) = \infty$$

Using the composite limit rule we obtain,

$$lim_{x\to \pi/2^-}\frac{sec(x)}{ln(sec(x))} = lim_{X\to\infty}\frac{X}{ln(X)}$$

We now use the hospital rule, the functions $h : x \mapsto x$ and $k : x \mapsto ln(x)$ are differentiable over $\mathbb{R}$, we have $lim_{x\to\infty}x = \infty$ and $lim_{x\to\infty}ln(x) = \infty$.
We notice that we have an undeterminate limit for the quotient ∞ over ∞. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $h'(x) = 1$and $k'(x) = 1/x$ by the Hospital rule we get

$$lim_{x\to \pi/2^-}\frac{sec(x)}{ln(sec(x))} = lim_{X\to\infty}\frac{X}{ln(X)} = lim_{X\to\infty}\frac{1}{1/X} = lim_{X\to\infty}X = \infty$$

9. $lim_{x\to\pi/2^-}(tan(x))^{sin(2x)}$

Solution :

$$(tan(x))^{sin(2x)} = e^{ln(tan(x))sin(2x)}$$

We now use the hospital rule, the functions $h : x \mapsto ln(tan(x))$ and $k : x \mapsto \frac{1}{sin(2x)}$ are differentiable over $(0, \pi/2)$, we have $lim_{x\to\pi/2^-}ln(tan(x)) = \infty$ and $lim_{x\to\pi/2^-}sin(2x) = 0^+$, since $0 \geqslant sin(2x)$ when $0 \leqslant x \leqslant \pi/2$, $lim_{x\to\pi/2^-}\frac{1}{sin(2x)} = \infty$.

We notice that we have an undeterminate limit for the quotient ∞ over ∞. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $h'(x) = \frac{sec^2(x)}{tan(x)} = \frac{sin(x)}{(cos(x))^3}$and $k'(x) = -\frac{2cos(2x)}{sin(2x)^2}$ by the Hospital rule we get

$$\begin{aligned}
lim_{x\to\pi/2^-}ln(tan(x))sin(2x) &= lim_{x\to\pi/2^-} - \frac{\frac{sin(x)}{(cos(x))^3}}{\frac{2cos(2x)}{sin(2x)^2}} \\
&= lim_{x\to\pi/2^-} - \frac{sin(x)sin(2x)^2}{2cos(2x)(cos(x))^3} \\
&= lim_{x\to\pi/2^-}\frac{sin(2x)^2}{2(cos(x))^3}
\end{aligned}$$

Since $x \mapsto \frac{sin(x)}{2cos(2x)}$ is continuous thus

$$lim_{x\to\pi/2^-}\frac{sin(x)}{2cos(2x)} = -2$$

$$\frac{sin(2x)^2}{2(cos(x))^3} = \frac{4sin(x)^2cos(x)^2}{2(cos(x))^3} = \frac{2sin(x)^2}{(cos(x))}$$

$$lim_{x\to\pi/2^-}cos(x) = 0^+$$

and

$$lim_{x\to\pi/2^-}2sin(x)^2 = 2$$

Then, by the quotient limit table we have,

$$lim_{x\to\pi/2^-}\frac{2sin(x)^2}{(cos(x))} = \infty$$

We have also

$$lim_{X\to\infty}e^X = \infty$$

Therefore, by compositing limits, we get

$$lim_{x\to\pi/2^-}(tan(x))^{sin(2x)} = \infty$$

10. $lim_{x\to\pi}\frac{\sqrt{1-tan(x)}-\sqrt{1+tan(x)}}{sin(2x)}$

Solution : We rationalize

$$\begin{aligned}\frac{\sqrt{1-tan(x)}-\sqrt{1+tan(x)}}{sin(2x)} &= \frac{1-tan(x)-(1+tan(x))}{sin(2x)(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})}\\ &= \frac{-2tan(x)}{sin(2x)(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})}\\ &= \frac{-2cos(x)}{2sin(x)^2cos(x)(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})}\\ &= \frac{-2}{2sin(x)^2(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})}\\ &= \frac{-1}{sin(x)^2(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})}\end{aligned}$$

$$lim_{x\to\pi}sin(x)^2(\sqrt{1-tan(x)}+\sqrt{1+tan(x)}) = 0^+$$

thus

$$lim_{x\to\pi}\frac{\sqrt{1-tan(x)}-\sqrt{1+tan(x)}}{sin(2x)} = lim_{x\to\pi}\frac{-1}{sin(x)^2(\sqrt{1-tan(x)}+\sqrt{1+tan(x)})} = -\infty$$

11. $lim_{x\to\infty}x^{1/ln(x)}$

    Solution :

    $$x^{1/ln(x)} = e^{ln(x^{1/ln(x)})} = e^{ln(x)/ln(x)} = e^1$$

    Thus

    $$lim_{x\to\infty}x^{1/ln(x)} = lim_{x\to\infty}e^1 = e^1$$

12. $lim_{x\to 0}cos(x)^{csc(x)}$

    Solution :

    $$cos(x)^{csc(x)} = e^{ln(cos(x))csc(x)} = e^{ln(cos(x))/sin(x)}$$

    We now use the hospital rule, the functions $f : x \mapsto ln(cos(x))$ and $g : x \mapsto sin(x)$ are differentiable at 0, we have $lim_{x\to 0}ln(cos(x)) = ln(cos(0)) = ln(1) = 0$ and $lim_{x\to 0}sin(x) = sin(0) = 0$.

    We notice that we have an undeterminate limit for the quotient 0 over 0. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $f'(x) = \frac{-sin(x)}{cos(x)}$ and $g'(x) = cos(x)$ by the Hospital rule we get

    $$lim_{x\to 0}\frac{ln(cos(x))}{sin(x)} = lim_{x\to 0}\frac{\frac{-sin(x)}{cos(x)}}{cos(x)} = lim_{x\to 0}\frac{-sin(x)}{cos(x)^2} = \frac{-sin(0)}{cos(0)^2} = 0$$

    Moreover

    $$lim_{X\to 0}e^X = e^0 = 1$$

    Then applying the limit composition rule, we

    $$lim_{x\to 0}cos(x)^{csc(x)} = 0$$

13. $lim_{x\to\pi/2}(\pi/2 - x)tan(x)$

    Solution :

We now use the hospital rule, the functions $f : x \mapsto \pi/2 - x$ and $g : x \mapsto \frac{cos(x)}{sin(x)}$ are differentiable over $(0, \pi)$, we have $lim_{x \to \pi/2} \pi/2 - x = 0$ and $lim_{x \to \pi/2} \frac{cos(x)}{sin(x)} = 0$.
We notice that we have an undeterminate limit for the quotient 0 over 0. Since all the condition required by the Hospital rule are satisfied we can apply it. Since we have $f'(x) = -1$and $g'(x) = \frac{1}{sin(x)^2}$, by the Hospital rule we get

$$\begin{aligned} lim_{x \to \pi/2} (\pi/2 - x) tan(x) &= lim_{x \to \pi/2} \frac{-1}{\frac{1}{sin(x)^2}} \\ &= lim_{x \to \pi/2} - sin(x)^2 \\ &= 0 \end{aligned}$$

Lots more limits can be found in tutorial 8 above is just a little selection.